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TABLE III.

Quintic.					
No.	**Name.**	**Aronhold and Clebsch Symbol.**	**Root Symbol.**	**Transvectant.**	**Remarks.**
1*	H	$(ab)^2a^3b^3$	$\Sigma(\alpha\beta)^2\gamma^2\delta^2\varepsilon^2$	$\{ff'\}^2$	
2*	S	$(ab)^4ab$	$\Sigma(\alpha\beta)^2(\gamma\delta)^2\varepsilon^2$	$\{ff'\}^4$	
3*	J	$(ab)^2(ac)a^2b^3c^4$	$\Sigma(\alpha\beta)^2(\alpha\gamma)(\beta\gamma^2\delta^3\varepsilon^3$	$\{fH\}$	
4	T	$(ab)^2(bc)^2(ac)^2abc$	$\Sigma(\alpha\beta)^2(\gamma\delta)^2(\alpha\gamma)(\beta\delta)\varepsilon^3$	$\{fH\}^4$	Clebsch's $\{Sf\}^2$
5*	C_1	$(ab)^4(ac)bc^4$	$\Sigma(\alpha\beta)^2(\gamma\delta)^2(\alpha\varepsilon)\beta\gamma\delta\varepsilon^2$	$\{Sf\}$	
6*	C_2	$(ab)^4(cd)^2(ac)bc^2d^3$	$\Sigma(\alpha\beta)^4(\gamma\delta)^2(\gamma\varepsilon)\gamma\delta^2\varepsilon^3$	$\{SH\}$	
7	C_3	$(ab)^4(cd)^2(ac)(bc)cd^3$	$\Sigma(\alpha\beta)^4(\gamma\delta)^2(\gamma\varepsilon)^2\delta^2\varepsilon^2$	$\{SH\}^2$	
8*	C_4	$(ab)^4(cd)^2(de)^2(ce)^2(ac)bde$	$\Sigma(\alpha\beta)^4(\gamma\delta)^4(\alpha\varepsilon)(\beta\varepsilon)(\gamma\varepsilon)\delta\varepsilon^2$	$\{ST\}$	Clebsch's $\{S^2f\}^3$
9*	α	$(ab)^4(cd)^2(de)^2(ce)^2(ac)(bd)e$	$\Sigma(\alpha\beta)^4(\gamma\delta)^4(\alpha\varepsilon)(\beta\varepsilon)(\gamma\varepsilon)(\delta\varepsilon)\varepsilon$	$\{ST\}^2$	Clebsch's $\{S^2f\}^4$
10*	A	$(ab)^4(cd)^4(ac)(bd)$	$\Sigma(\alpha\beta)^4(\gamma\delta)^2(\gamma\varepsilon)^2(\delta\varepsilon)^2$	$\{SS'\}^2$	See foot-note, p. 114. Salmon's J invariant
11	τ	$(ab)^4(cd)^4(ef)^2(ae)(be)(cf)(df)ef$	$\Sigma(\alpha\beta)^6(\gamma\delta)^4(\gamma\varepsilon)^2(\delta\varepsilon)^2\varepsilon^2$	$\{TT'\}^2$	Clebsch's $\{S^2H\}^4$
12*	β	$(ab)^4(cd)^4(ef)^2(fl)^2(el)^2(ce)(df)(al)b$	$\Sigma(\alpha\beta)^6(\gamma\delta)^4(\delta\varepsilon)^3(\gamma\varepsilon)^2(\alpha\varepsilon)(\beta\varepsilon)\gamma$	$\{aS\}$	Clebsch's $\{S^3f\}^5$
13*	C_5	$(ab)^4(cd)^2(ac)(bc)(ef)^4(ec)d^3f$	$\Sigma(\alpha\beta)^6(\gamma\delta)^4(\gamma\varepsilon)^2(\delta\varepsilon)\delta\varepsilon^3$	$\{C_3S\}$	
14	C_6	$(ab)^4(cd)^2(ac)(bc)(ce)d^3e^4$	$\Sigma(\alpha\beta)^4(\gamma\delta)^2(\gamma\varepsilon)^2(\alpha\delta)\beta\gamma\delta^2\varepsilon^3$	$\{C_3f\}$	Clebsch's $\{SJ\}^2$
15	C_7	$(ab)^2(bc)^2(ac)^2(de)^2(ef)^2(df)^2(ad)(be)(cm)fm^4$	$\Sigma(\alpha\beta)^6(\gamma\delta)^4(\gamma\varepsilon)^3(\delta\varepsilon)^2a\beta\delta\varepsilon^2$	$\{f\tau\}$	

* A unique form.

TABLE III—Continued.

Quintic.					
No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.	Remarks.
16	C_8	$(ab)^2(bc)^2(ac)^2(de)^2(ef)^2(df)^2(ad)(be)(lm)^2(mo)^2(lo)^2(el)(fmo$	$\Sigma(\alpha_1\beta)^8(\gamma\delta)^4(\gamma\varepsilon)^4(\delta\varepsilon)^3(\alpha_1\gamma)(_1\beta\varepsilon)\delta^2\varepsilon$	$\{T\tau\}$	
17	γ	$(ab)^2(bc)^2(ac)^2(de)^2(ef)^2(df)^2(ad)(be)(lm)^4(or)^2(rt)^2(ot)^2(lo)(mr)(ct)(_1f$	$\Sigma(\alpha_1\beta)^{10}(\gamma\delta)^6(\gamma\varepsilon)^5(\delta\varepsilon)^4(a\varepsilon)(_1\beta\varepsilon)\delta$	$\{a\tau\}$	
18	L	$(ab)^2(bc)^2(ac)^2(de)^2(ef)^2(df)^2(ad)(bc)(lm)^2(mo)^2(lo)^2(rs)^2(rn)^2(sn)^2(lr)(ms)(cs)(_1fn)$	$\Sigma(\alpha_1\beta)^8(\gamma\delta)^4(a\gamma)^4(\gamma\varepsilon)^4(\delta\varepsilon)^6(_1\beta\delta)^2(_1\beta\varepsilon)^2$	$\{\tau\tau'\}^2$	Compare with Faà de Bruno's I_{12}
19*	ϑ	$(ab)^4(cd)^2(ce)^2(de)^2(_1fl)^2(_1fn)^2(ln)^2(cf)(dl)(ae)bn$	$\Sigma(\alpha_1\beta)^8(\gamma\delta)^4(\gamma\varepsilon)^4(\delta\varepsilon)^3\delta\varepsilon$	$\{S\tau\}$	
20	K	$(ab)^4(cd)^2(ce)^2(de)^2(_1fl)^2(_1fn)^2(ln)^2(cf)(dl)(ae)(bn)$	$\Sigma(\alpha_1\beta)^8(\gamma\delta)^4(\gamma\varepsilon)^4(\delta\varepsilon)^4$	$\{S\tau\}^2$	Faà de Bruno's I_8 is another expression for K.
21	δ	$(ab)^4(cd)^2(ce)^2(de)^2(_1fl)^2(_1fn)^2(ln)^2(cf)(dl)(ae)(oq)^4(rs)^2(rt)^2(or)(qs)(bt)n$	$\Sigma(\alpha_1\beta)^{12}(\gamma\delta)^8(a\varepsilon)(_1\beta\varepsilon)(\gamma\varepsilon)^5(\delta\varepsilon)^5\varepsilon$	$\{\vartheta a\}$	
22	I	$(ab)^2(bc)^2(ac)^2(de)^2(ef)^2(df)^2(ad)(be)(lm)^4(or)^2(rt)^2(ot)^2(lo)(mr)(ct)(hi)^4(jk)^4(uv)^2(vw)^2(uw)^2(_1ju)(kv)(hw)(_1fi)$	$\Sigma(\alpha_1\beta)^{16}(\gamma\delta)^{11}(\gamma\varepsilon)^7(\delta\varepsilon)^7(a\varepsilon)^2(_1\beta\varepsilon)^2$	$\{\beta\tau\}$	

* A unique form.

TABLE IV.

Sextic.

No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.	Remarks.
1*	H	$(ab)^2a^4b^4$	$\Sigma(a_i\beta)^2\gamma^2\delta^2\varepsilon^2\lambda^2$	$\{f_xf'_x\}^2$	
2*	S	$(ab)^4a^2b^2$	$\Sigma(a_i\beta)^2(\gamma\delta)^2\varepsilon^2\lambda^2$	$\{f_xf'_x\}^4$	
3*	J	$(ab)^2(ac)a^3b^4c^5$	$\Sigma(a_i\beta)^2(\alpha\gamma)_i\beta\gamma^2\delta^3\varepsilon^3\lambda^3$	$\{f_xH\}$	
4*	A_6	$(ab)^6$	$\Sigma(a_i\beta)^2(\gamma\delta)^2(\varepsilon\lambda)^2$	$\{f_xf'_x\}^6$	
5*	C_1	$(ab)^4(ac)ab^2c^5$	$\Sigma(a_i\beta)^2(\gamma\delta)^2(\alpha\varepsilon)_i\beta\gamma\delta\varepsilon^2\lambda^3$	$\{Sf_x\}$	
6	C_2	$(ab)^4(ac)^2b^2c^4$	$\Sigma(a_i\beta)^2(\gamma\delta)^2(\alpha\varepsilon)(_i\beta\varepsilon)\gamma\delta\varepsilon\lambda^3$	$\{Sf_x\}^2$	Clebsch's p
7*	l	$(ab)^4(ac)^2(bc)^2c^2$	$\Sigma(a_i\beta)^2(\gamma\delta)^2(\alpha\varepsilon)(_i\beta\varepsilon)(\gamma\lambda)(\delta\lambda)\varepsilon\lambda$	$\{Sf_x\}^4$	
8*	C_3	$(ab)^4(cd)^2(ac)ab^2c^3d^4$	$\Sigma(a_i\beta)^4(\gamma\delta)^2(\gamma\varepsilon)\gamma\delta^2\varepsilon^3\lambda^4$	$\{HS\}$	See C_3 of Quintic
9	B	$(ab)^4(cd)^4(ac)^2(bd)^2$	$\Sigma(a_i\beta)^4(\gamma\delta)^4(\varepsilon\lambda)^4$	$\{SS'\}^4$	
10*	C_a	$(ab)^4(ac)^2(bc)^2(cd)cd^5$	$\Sigma(a_i\beta)^2(\gamma\delta)^2(\alpha\varepsilon)^2(_i\beta\varepsilon)(\gamma\lambda)(\delta\lambda)_i\beta\gamma\delta\varepsilon\lambda^2$	$\{f'l\}$	
11	C_b	$(ab)^4(ac)^2(bc)^2(cd)^2d^4$	$\Sigma(a_i\beta)^2(\gamma\delta)^2(\alpha\varepsilon)^2(_i\beta\varepsilon)(\gamma\lambda)^2(\delta\lambda)_i\beta\delta\varepsilon\lambda$	$\{f'l\}^2$	
12	C_5	$(ab)^4(ac)^2(bc)^2(de)^2(cd)cd^3e^4$	$\Sigma(a_i\beta)^4(\gamma\delta)^2(\alpha\varepsilon)(_i\beta\varepsilon)(\gamma\lambda)^2(\delta\lambda)\gamma\delta^2\varepsilon^3\lambda^2$	$\{Hl\}$	See C_5 of Quintic
13*	C_c	$(ab)^4(ac)^2(bc)^2(de)^4(cd)cde^2$	$\Sigma(a_i\beta)^4(\gamma\delta)^4(\alpha\varepsilon)(_i\beta\varepsilon)(\gamma\lambda)(\delta\lambda)(\varepsilon\lambda)\varepsilon^2\lambda^2$	$\{Sl\}$	
14	m	$(ab)^4(ac)^2(bc)^2(de)^4(cd)^2c^2$	$\Sigma(a_i\beta)^4(\gamma\delta)^4(\alpha\varepsilon)(_i\beta\varepsilon)(\gamma\lambda)(\delta\lambda)(\varepsilon\lambda)^2\varepsilon\lambda$	$\{Sl\}^2$	
15	C	$(ab)^4(ac)^2(bc)^2(de)^4(df')^2(ef')^2(cf)^2$	$\Sigma(a_i\beta)^4(\gamma\delta)^4(\varepsilon\lambda)^2(\alpha\varepsilon)(_i\beta\varepsilon)(\gamma\lambda)(\delta\lambda)$	$\{ll'\}^2$	
16*	λ	$(ab)^4(ac)^2(bc)^2(de)^4(cd)^2(_xf'l)^4(_xfu)^2(lu)^2(eu)eu$	$\Sigma(a_i\beta)^6(\gamma\delta)^6(\alpha\varepsilon)^2(_i\beta\varepsilon)^2(\gamma\lambda)^2(\delta\lambda)^2(\varepsilon\lambda)^3\varepsilon\lambda$	$\{ml\}$	

* A unique form.

TABLE IV—Continued.

Sextic.					
No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.	Remarks.
17	C_d	$(ab)^4(ac)^2(bc)^2(de)^4(df)^2(ce)cef^4$	$\Sigma(\alpha_1\beta)^4(\gamma\delta)^4(a\varepsilon)^2({}_1\beta\varepsilon)^2(\gamma\varepsilon)(\gamma\lambda)(\delta\lambda)\delta\varepsilon\lambda^4$	$\{C_2l\}$	Compare order and degree of 17 and 18
18	C_{dd}	$(ab)^4(ac)(de)^4(df)^2(ef)^2(bf)^2ac^5$	$\Sigma(\alpha_1\beta)^4(\gamma\delta)^4(a\varepsilon)^2({}_1\beta\varepsilon)^2(\gamma\lambda)^2(\delta\lambda)\delta\varepsilon^2\lambda^3$	$\{C_1l\}^2$	
19	C_e	$(ab)^4(ac)^2(bc)^2(cd)^2(ef)^4(el)^2(fl)^2(ld)ld^3$	$\Sigma(\alpha_1\beta)^4(\gamma\delta)^4(a\varepsilon)^3({}_1\beta\varepsilon)^2(\gamma\lambda)^3(\delta\lambda)^3{}_1\beta\varepsilon^2\lambda$	$\{C_bl\}$	
20	n	$(ab)^4(ac)^2(bc)^2(de)^4(cd)^2(ln)^4(el)^2n^2$	$\Sigma(\alpha_1\beta)^6(\gamma\delta)^6(a\varepsilon)({}_1\beta\varepsilon)(\gamma\lambda)(\delta\lambda)(\varepsilon\lambda)^4\varepsilon\lambda$	$\{Sm\}^2$	
21	μ	$(ab)^4(ac)^2(bc)^2(de)^4(cd)^2(ln)^4(el)^2(rs)^4(rt)^2(st)^2(nt)nt$	$\Sigma(\alpha_1\beta)^8(\gamma\delta)^8(a\varepsilon)^2({}_1\beta\varepsilon)^2(\gamma\lambda)^2(\delta\lambda)^2(\varepsilon\lambda)^5\varepsilon\lambda$	$\{nl\}$	
22	ν	$(ab)^4(ac)^2(bc)^2(de)^4(cd)^2(ln)^4(el)^2(on)^4(or)^2(nr)^2(st)^4(rs)^2(nt)nt$	$\Sigma(\alpha_1\beta)^{10}(\gamma\delta)^{10}(a\varepsilon)^2({}_1\beta\varepsilon)^2(\gamma\lambda)^2(\delta\lambda)^2(\varepsilon\lambda)^7\varepsilon\lambda$	$\{mn\}$	
23	C_{ee}	$(ab)^4(ac)^2(bc)^2(de)^4(cd)^2(lm)^4(ln)^2(mn)^2(mo)^2(eo)eo^3$	$\Sigma(\alpha_1\beta)^6(\gamma\delta)^6(a\varepsilon)^3({}_1\beta\varepsilon)^2(\gamma\lambda)^3(\delta\lambda)^2(\varepsilon\lambda)^3{}_1\beta\delta\varepsilon\lambda$	$\{C_bm\}$	
24	D	$(ab)^4(ac)^2(bc)^2(de)^4(cd)^2(lm)^4(ln)^2(mn)^2(op)^4(no)^2(ep)^2$	$\Sigma(\alpha_1\beta)^8(\gamma\delta)^8(\varepsilon\lambda)^6(a\varepsilon)^2({}_1\beta\varepsilon)^2(\gamma\lambda)^2(\delta\lambda)^2$	$\{mm'\}^2$	
25	E	$(ab)^4(ac)^2(bc)^2(de)^4(cd)^2(fl)^4(fn)^2(ln)^2(en)(op)^4(oq)^2(pq)^2(rs)^4(qr)^2(tv)^4(st)^2(ve)(vn)$	$\Sigma(\alpha_1\beta)^{12}(\gamma\delta)^{12}(\varepsilon\lambda)^9(a\varepsilon)^3({}_1\beta\varepsilon)^3(\gamma\lambda)^3(\delta\lambda)^3$	$\{\lambda n\}^2$	

* A unique form.

TABLE V.

No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.	Remarks.
		Linear and Linear.			
1*	I	(aa')	(aa')	$\{f_{(1)}, f_{(2)}\}$	
		Linear and Quadric.			
1*	C	$(aa')a$	$\Sigma(aa')_1\beta$	$\{f_{(1)}, f_{(2)}\}$	
2*	I	$(aa')(ab')$	$(aa')(_1\beta a')$	$\{Cf_{(1)}\}$	
		Linear and Cubic.			
1*	C_1	$(aa')a^2$	$\Sigma(aa')_1\beta\gamma$	$\{f_{(1)}, f_{(2)}\}$	
2*	C_2	$(ab)^2(aa')b$	$\Sigma(a_1\beta)^2(\gamma a')\gamma$	$\{H_{(2)}, f_{(1)}\}$	Also $\{C_1, f_{(2)}\}^2$
3*	C_3	$(ab)^2(ac)(ba')c^2$	$\Sigma(a_1\beta)^2(a\gamma)(_1\beta a')\gamma^2$	$\{J_{(2)}, f_1^3\}$	Also $\{C_2, f_{(2)}\}$
4*	C_4	$(aa')(ab')a$	$\Sigma(aa')(_1\beta a')\gamma$	$\{C_1 f_{(1)}\}$	See No. 2 of Linear and Quadric
5*	I_1	$(ab)^2(aa')(bb')$	$\Sigma(a_1\beta)^2(\gamma a')^2$	$\{C_2, f_{(1)}\}$	Also $\{C_1 C_1'\}^2$
6*	C_5	$(ab)^2(ac)(ba')(cb')c$	$\Sigma(a_1\beta)^2(a\gamma)(a'\gamma)^2{}_1\beta$	$\{C_3, f_{(1)}\}$	Also $\{C_1 C_2\}$
7*	I_2	$(aa')(ab')(ac')$	$(aa')(_1\beta a')(\gamma a')$	$\{C_4, f_{(1)}\}$	
8*	I_3	$(ab)^2(ac)(ba')(cb')(cc')$	$\Sigma(a_1\beta)^2(a\gamma)(a'_1\beta)(a'\gamma)^2$	$\{C_5, f_{(1)}\}$	Also $\{C_2 C_4\}$

* A unique form.

TABLE VI.

LINEAR AND QUARTIC.

No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.	Remarks.
1*	C_1	$(aa')a^3$	$\Sigma(aa')_1\beta\gamma\delta$	$\{f_{(1)}f_{(2)}\}$	
2*	C_2	$(aa')(ab)^2ab^2$	$\Sigma(a_1\beta)^2(\gamma a')\gamma\delta^2$	$\{f_{(1)}H_{(2)}\}$	Also $\{C_1 f_{(2)}\}^2$
3	C_3	$(ab)^2(ac)(aa')b^2c^3$	$\Sigma(a_1\beta)^2(a\gamma)(_1\beta a')\gamma^2\delta^3$	$\{f_{(1)}J_{(2)}\}$	Also $\{C_2 f_{(2)}\}$
4*	C_4	$(aa')(ab')a^2$	$\Sigma(aa')(_1\beta a')\gamma\delta$	$\{f_{(1)}C_1\}$	
5	C_5	$(ab)^2(aa')(ab')b^2$	$\Sigma(a_1\beta)^2(\gamma a')^2\delta^2$	$\{f_{(1)}C_2\}$	Also $\{C_1C_1'\}^2$
6	C_6	$(ab)^2(ac)(ca')(cb')ab^2c$	$\Sigma(a_1\beta)^2(aa')(\gamma a')(_1\beta\gamma)\gamma\delta^3$	$\{C_1C_2\}$	Also $\{f_{(1)}C_3\}$
7*	C_7	$(aa')(ab')(ac')a$	$\Sigma(aa')(_1\beta a')(\gamma a')\delta$	$\{f_{(1)}C_4\}$	
8*	C_8	$(ab)^2(aa')(ab')(bc')b$	$\Sigma(a_1\beta)^2(\gamma a')(\delta a')\delta$	$\{f_{(1)}C_5\}$	Also $\{C_1C_4\}^2$
9	C_9	$(ab)^2(ac)(ca')(cb')(cc')ab^2$	$\Sigma(a_1\beta)^2(_1\beta\delta)(aa')(\gamma a')^2\gamma\delta^2$	$\{C_1C_5\}$	Also $\{C_7H_{(2)}\}$
10	C_{10}	$(ab)^2(bc)(aa')(ab')(bc')(cd')c^2$	$\Sigma(a_1\beta)^2(a\gamma)(_1\beta a')(\delta a')^3\gamma^2$	$\{C_1C_8\}$	Also $\{C_4C_5\}$
11*	I_1	$(aa')(ab')(ac')(ad')$	$(aa')(_1\beta a')(\gamma a')(\delta a')$	$\{f_{(1)}C_7\}$	
12*	I_2	$(ab)^2(aa')(ab')(bc')(bd')$	$\Sigma(a_1\beta)^2(\gamma a')^2(\delta a')^2$	$\{f_1C_8\}$	Also $\{C_4C_4'\}^2$
13	C_{11}	$(ab)^2(bc)(aa')(ab')(bc')(cd')(ce')c$	$\Sigma(a_1\beta)^2(a\gamma)(_1\beta a')(\gamma a')^2(\delta a')^2\delta$	$\{C_5C_7\}$	Also $\{C_4C_8\}$
14*	I_3	$(ab)^2(bc)(aa')(ab')(bc')(cd')(ce')(cf')$	$\Sigma(a_1\beta)^2(a\gamma)(_1\beta a')(\gamma a')^2(\delta a')^3$	$\{f_{(1)}C_{11}\}$	

* Uniqne form.

TABLE VII.

No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.	Remarks.
		Quadric and Quadric.			
1*	J	$(aa')aa'$	$\Sigma(aa')_1\beta_1\beta'$	$\{f_{(1)}f_{(2)}\}$	The Cayley Symbol is $\overline{12}$
2*	Δ	$(aa')^2$	$\Sigma(aa')(_1\beta_1\beta')$	$\{f_{(1)}f_{(2)}\}^2$	The Cayley Symbol is $\overline{12}^2$
		Quadric and Cubic.			
1*	ϑ	$(aa')aa'^2$	$\Sigma(aa')_1\beta_1\beta'\gamma'$	$\{f_{(1)}f_{(2)}\}$	See J of Quadric and Quadric
2*	L_1	$(aa')^2a'$	$\Sigma(aa')(_1\beta_1\beta')\gamma'$	$\{f_{(1)}f_{(2)}\}^2$	See Δ of Quadric and Quadric
3*	Ω	$(a'b')^2(aa')ab'$	$\Sigma(a'_1\beta')^2(a\gamma')_1\beta\gamma'$	$\{f_{(1)}H_{(2)}\}$	
4*	I	$(a'b')^2(aa')(ab')$	$\Sigma(a'_1\beta')^2(a\gamma')(_1\beta\gamma')$	$\{f_{(1)}H_{(2)}\}^2$	
5*	L_2	$(aa')^2(ba')b$	$\Sigma(aa')(_1\beta_1\beta')(a\gamma')_1\beta$	$\{L_1f_{(1)}\}$	
6*	L_3	$(a'b')^2(a'c')(ac')^2b'$	$\Sigma(a'\gamma')^2(_1\beta'_1\gamma')(aa')(_1\beta_1\beta')_1\beta'$	$\{L_1H_{(2)}\}$	
7	L_4	$(a'b')^2(a'c')(ac')^2(bb')b$	$\Sigma(a'\gamma')^2(_1\beta'\gamma')(aa')(_1\beta_1\beta')^2a$	$\{L_3f_{(1)}\}$	
8	R	$(aa')^2(bb')^2(ca')(cb')$	$\Sigma(aa')^2(_1\beta_1\beta')^2(a\gamma')(_1\beta\gamma')$	$\{L_1L_2\}$	
9*	M	$(a'b')^2(a'c')(ac')^2(bd')^2(cd')(cb')$	$\Sigma(a'_1\beta')^2(\beta'\gamma')(aa')^2(a_1\beta')(_1\beta\gamma')^3$	$\{L_2L_3\}$	

* A unique form.

TABLE VIII.

QUADRIC AND QUARTIC.

No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.	Remarks.
1*	L	$(aa')aa'^3$	$\Sigma(aa')_1\beta_1\beta'\gamma'\delta'$	$\{f_{(1)}f_{(2)}\}$	
2*	ψ	$(aa')^2a'^2$	$\Sigma(aa')(_1\beta_1\beta')\gamma'\delta'$	$\{f_{(1)},f_{(2)}\}^2$	
3*	Ψ	$(aa')^2(ba')a'b$	$\Sigma(aa')(_1\beta_1\beta')(\alpha\gamma')_1\beta\delta'$	$\{\psi,f_{(1)}\}$	
4*	φ	$(aa')^2(ba')^2$	$\Sigma(aa')(_1\beta_1\beta')(\alpha\gamma')(_1\beta\delta')$	$\{\psi,f_{(1)}\}^2$	Clebsch's A
5*	M	$(a'b')^2(aa')aa'b'^2$	$\Sigma(a'_1\beta')^2(\alpha\gamma')_1\beta\gamma'\delta'^2$	$\{f_{(1)}H_{(2)}\}$	
6	χ	$(a'b')^2(aa')^2b'^2$	$\Sigma(a'_1\beta')^2(\alpha\gamma')(_1\beta\gamma')\delta'^2$	$\{f_{(1)}H_{(2)}\}^2$	
7*	X	$(a'b')^2(aa')^2(bb')bb'$	$\Sigma(a'_1\beta')^2(\alpha\gamma')^2(_1\beta\delta')_1\beta\delta'$	$\{\chi f_{(1)}\}$	
8	Σ	$(a'b')^2(aa')^2(bb')^2$	$\Sigma(a'_1\beta')^2(\alpha\gamma')^2(_1\beta\delta')^2$	$\{\chi,f_{(1)}\}^2$	Clebsch's B
9	τ	$(a'b')^2(aa')^2(bc')^2(b'c')b'c'$	$\Sigma(a'_1\beta')^2(\alpha\gamma')^2(_1\beta_1\beta')(_1\beta\gamma')(a'\delta')\delta'^2$	$\{\psi\chi\}$	
10*	C	$(a'b')^2(aa')^2(bc')^2(b'c')(cb')(cc')$	$\Sigma(a'_1\beta')^2(a_1\beta)(\alpha\gamma')^3(_1\beta\delta')^3$	$\{\tau f_{(1)}\}^2$	
11	K	$(a'b')^2(aa')^2(b'c')b'c'^3$	$\Sigma(a'_1\beta')^2(\alpha\gamma')(_1\beta\gamma')(a'\delta')_1\beta\gamma'\delta'^2$	$\{\chi f_{(2)}\}$	Clebsch's $\{\psi H_2\}$

* A unique form.

TABLE IX.

Cubic and Cubic.

No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.	Remarks.
1*	J	$(aa')a^2a'^2$	$\Sigma(aa')_1\beta_1\beta'\gamma\gamma'$	$\{f_{(1)}f_{(2)}\}$	
2*	θ	$(aa')^2aa'$	$\Sigma(aa')(_1\beta_1\beta')\gamma\gamma'$	$\{f_{(1)}f_{(2)}\}^2$	
3*	P	$(aa')^3$	$\Sigma(aa')(_1\beta_1\beta')(\gamma\gamma')$	$\{f_{(1)}f_{(2)}\}^3$	Clebsch's J
4	C_1	$(a'b')^2(aa')a^2b'$	$\Sigma(a'_1\beta')^2(\alpha\gamma')_1\beta\gamma\gamma'$	$\{H_{(2)}f_{(1)}\}$	
5*	π	$(a'b')^2(aa')(ab')a$	$\Sigma(a'_1\beta')^2(\alpha\gamma')(_1\beta\gamma')\gamma$	$\{H_{(2)}f_{(1)}\}^2$	
6	C_2	$(ab)^2(aa')ba'^2$	$\Sigma(a_1\beta)^2(\gamma\alpha')_1\beta'\gamma\gamma'$	$\{H_{(1)}f_{(2)}\}$	
7*	p	$(ab)^2(aa')(ba')a'$	$\Sigma(a_1\beta)^2(\gamma\alpha')(_1\beta'\gamma')\gamma$	$\{H_{(1)}f_{(2)}\}^2$	
8	C_3	$(ab)^2(a'b')^2(aa')bb'$	$\Sigma(a_1\beta)^2(a'_1\beta')^2(\gamma\gamma')\gamma\gamma'$	$\{H_{(1)}H_{(2)}\}$	
9	N	$(ab)^2(a'b')^2(aa')(bb')$	$\Sigma(a_1\beta)^2(a'_1\beta')^2(\gamma\gamma')^2$	$\{H_{(1)}H_{(2)}\}^2$	Clebsch's T'
10*	C_4	$(a'b')^2(c'd')^2(aa')(ab')(ac')d'$	$\Sigma(a'_1\beta')^4(\alpha\gamma')(_1\beta\gamma')(\gamma\gamma')\gamma'$	$\{H^2_{(2)}f_{(1)}\}^3$	
11*	C_5	$(ab)^2(cd)^2(aa')(ba')(ca')d$	$\Sigma(a_1\beta)^4(\gamma\alpha')(\gamma_1\beta')(\gamma\gamma')\gamma$	$\{H^2_{(1)}f_{(2)}\}^3$	
12	C_6	$(a'b')^2(a'c')(ac')^2ab'$	$\Sigma(a'_1\beta')^2(\alpha'\gamma')(a_1\beta')(_1\beta\gamma')\gamma\gamma'$	$\{J_{(2)}f_{(1)}\}^2$	
13*	M	$(a'b')^2(a'c')(ac')^2(ab')$	$\Sigma(a'_1\beta')^2(\alpha'\gamma')(a_1\beta')(_1\beta\gamma')(\gamma\gamma')$	$\{J_{(2)}f_{(1)}\}^3$	
14	C_7	$(ab)^2(ac)(ca')^2ba'$	$\Sigma(a_1\beta)^2(\alpha\gamma)(_1\beta a')(\gamma_1\beta')\gamma\gamma'$	$\{J_{(1)}f_{(2)}\}^2$	
15*	M'	$(ab)^2(ac)(ca')^2(ba')$	$\Sigma(a_1\beta)^2(\alpha\gamma)(_1\beta a')(\gamma_1\beta')(\gamma\gamma')$	$\{J_{(1)}f_{(2)}\}^3$	
16	Q	$(ab)^2(ac)(a'b')^2(a'c')(bb')(cc')^2$	$\Sigma(a_1\beta)^2(\alpha\gamma)(a'_1\beta')^2(a'\gamma')(_1\beta_1\beta')(\gamma\gamma')^2$	$\{J_{(1)}J_{(2)}\}^3$	
17	C_8	$(ab)^2(a'b')^2(a'c')(ab')(bc')c'$	$\Sigma(a_1\beta)^2(a'_1\beta')^2(a'\gamma')(\gamma\gamma')^2{}_1\beta'$	$\{H_{(1)}J_{(2)}\}^2$	
18	C_9	$(a'b')^2(ab)^2(ac)(ba')(cb')c$	$\Sigma(a'_1\beta')^2(a_1\beta)^2(\alpha\gamma)(_1\beta\gamma')(\gamma\gamma')^2{}_1\beta$	$\{H_{(2)}J_{(1)}\}^2$	

* A unique form.

TABLE X.

Cubic and Quartic.

No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.
1*	C_1	$(aa')a^2a'^3$	$\Sigma(aa')_1\beta\gamma_1\beta'\gamma'\delta'$	$\{f_{(1)}f_{(2)}\}$
2*	C_2	$(aa')^2aa'^2$	$\Sigma(aa')(_1\beta_1\beta')\gamma\gamma'\delta'$	$\{f_{(1)}f_{(2)}\}^2$
3*	p	$(aa')^3a'$	$\Sigma(aa')(_1\beta_1\beta')(\gamma\gamma')\delta'$	$\{f_{(1)}f_{(2)}\}^3$
4	C_3	$(ab)^2(aa')ba'^3$	$\Sigma(a_1\beta)^2(\gamma a')\gamma_1\beta'\gamma'\delta'$	$\{H_{(1)}f_{(2)}\}$
5	C_4	$(ab)^2(aa')(ba')a'^2$	$\Sigma(a_1\beta)^2(\gamma a')(\gamma_1\beta')\gamma'\delta'$	$\{H_{(1)}f_{(2)}\}^2$
6	C_5	$(ab)^2(ac)(ca')^2ba'^2$	$\Sigma(a_1\beta)^2(a\gamma)(_1\beta a')(\gamma_1\beta')\gamma\gamma'\delta'$	$\{J_{(1)}f_{(2)}\}^2$
7	s	$(ab)^2(ac)(ba')(ca')^2a'$	$\Sigma(a_1\beta)^2(a\gamma)(_1\beta a')(\gamma_1\beta')(\gamma\gamma')\delta'$	$\{J_{(1)}f_{(2)}\}^3$
8	C_6	$(a'b')^2(aa')a^2a'b'^2$	$\Sigma(a'_1\beta')^2(a\gamma')_1\beta\gamma'\delta'^2$	$\{H_{(2)}f_{(1)}\}$
9	C_7	$(a'b')^2(aa')^2ab'^2$	$\Sigma(a'_1\beta')^2(a\gamma')(_1\beta\gamma')\gamma\delta'^2$	$\{H_{(2)}f_{(1)}\}^2$
10*	π	$(a'b')^2(aa')^2(ab')b'$	$\Sigma(a'_1\beta')^2(a\gamma')(_1\beta\gamma')(\gamma\delta')\delta'$	$\{H_{(2)}f_{(1)}\}^3$
11	C_8	$(ab)^2(a'b')^2(aa')ba'b'^2$	$\Sigma(a_1\beta)^2(a'_1\beta')^2(\gamma\gamma')\gamma\gamma'\delta'^2$	$\{H_{(1)}H_{(2)}\}$
12	C_9	$(ab)^2(a'b')^2(aa')(bb')a'b'$	$\Sigma(a_1\beta)^2(a'_1\beta')^2(\gamma\gamma')^2\delta'^2$	$\{H_{(1)}H_{(2)}\}^2$
13	C_{10}	$(ab)^2(a'b')^2(a'c')(aa')(bb')b'c'^3$	$\Sigma(a_1\beta)^2(a'_1\beta')^2(a'\gamma')(\gamma\gamma')^2{}_1\beta\delta'^3$	$\{H_{(1)}J_{(2)}\}^2$
14	C_{11}	$(a'b')^2(a'c')(ac')^3a'b'^2$	$\Sigma(a'_1\beta')^2(a'\gamma')(a_1\beta')(_1\beta\gamma')(\gamma\gamma')\delta'^3$	$\{f_{(1)}J_{(2)}\}^3$
15	C_{12}	$(ab)^2(ac)(a'b')^2(ca')^2bb'^2$	$\Sigma(a_1\beta)^2(a\gamma)(a'_1\beta')^2(\gamma\gamma')^2{}_1\beta\delta'^2$	$\{H_{(2)}J_{(1)}\}^2$
16	σ	$(ab)^2(ac)(a'b')^2(bb')(ca')^2b'$	$\Sigma(a_1\beta)^2(a\gamma)(a'_1\beta')^2(\gamma\gamma')^2(_1\beta\delta')\delta'$	$\{H_{(2)}J_{(1)}\}^3$
17	C_{13}	$(ab)^2(ac)(a'b')^2(a'c')(bb')(cc')^2a'b'c'$	$\Sigma(a_1\beta)^2(a\gamma)(a'_1\beta')^2(a'\gamma')(_1\beta\gamma')(\gamma\gamma')^2\delta'^3$	$\{J_{(1)}J_{(2)}\}^3$

* A unique form.

18	C_{14}	$(aa')^3(ba')b^2$	$\Sigma(a\alpha')(_i\beta\beta')(\gamma\gamma')(a\delta')_i\beta\gamma$	$\{f_{(1)}p\}$
19	q	$(ab)^2(aa')(ca')^3b$	$\Sigma(a_i\beta)^2(a\alpha')(_i\beta\beta')(\gamma\gamma')(\gamma\delta')\gamma$	$\{H_{(1)}p\}$
20	C_{15}	$(ab)^2(ac)(ba')(da')^3c^2$	$\Sigma(a_i\beta)^2(a\gamma)(a\alpha')(_i\beta\beta')(\gamma\gamma')(_i\beta\delta')\gamma^2$	$\{J_{(1)}p\}$
21	C_{16}	$(ab)^2(cd)^2(aa')(ba')(ca')da'$	$\Sigma(a_i\beta)^4(\gamma\alpha')(\gamma_i\beta')(\gamma\gamma')\gamma\delta'$	$\{C_4H_{(1)}\}$
22*	B	$(ab)^2(cd)^2(aa')(ba')(ca')(da')$	$\Sigma(a_i\beta)^4(\gamma\alpha')(\gamma_i\beta')(\gamma\gamma')(\gamma\delta')$	$\{C_4H_{(1)}\}^2$
23*	C_{17}	$(ab)^2(ac)(de)^2(ba')(ca')^2(da')e$	$\Sigma(a_i\beta)^4(a\gamma)(_i\beta\alpha')(\gamma_i\beta')(\gamma\gamma')(\gamma\delta')\gamma$	$\{H_{(1)}s\}$
24	C_{18}	$(a'b')^2(aa')^2(ab')(bb')b^2$	$\Sigma(a'_i\beta')^2(a\gamma')(_i\beta\gamma')(\gamma\delta')^2a_i\beta$	$\{f_{(1)}\pi\}$
25	C_{19}	$(ab)^2(a'b')^2(ca')^2(ab')(cb')b$	$\Sigma(a_i\beta)^2(a'_i\beta')^2(a\gamma')(_i\beta\gamma')(\gamma\delta')^2\gamma$	$\{H_{(1)}\pi\}$
26	C_{20}	$(ab)^2(ac)(a'b')^2(da')(bb')(db')c^2$	$\Sigma(a_i\beta)^2(a\gamma)(a'_i\beta')^2(a\gamma')(_i\beta\gamma')(\gamma\delta')^2{}_i\beta\gamma$	$\{J_{(1)}\pi\}$
27	C_{21}	$(ab)^2(ac)(de)^2(a'b')^2(ca')^2(bb')(db')e$	$\Sigma(a_i\beta)^4(a\gamma)(a'_i\beta')^2(\gamma\gamma')^2(_i\beta\delta')(\gamma\delta')\gamma$	$\{H_{(1)}\sigma\}$
28	C_{22}	$(ab)^2(cd)^2(a'b')^2(aa')(ca')(bb')db'$	$\Sigma(a_i\beta)^4(a'_i\beta')^2(\gamma\gamma')^2(\gamma\delta')\gamma\delta'$	$\{C_9H_{(1)}\}$
29	D	$(ab)^2(cd)^2(a'b')^2(aa')(ca')(bb')(db')$	$\Sigma(a_i\beta)^4(a'_i\beta')^2(\gamma\gamma')^2(\gamma\delta')^2$	$\{C_9H_{(1)}\}^2$
30	C_{23}	$(ab)^2(a'b')^2(a'c')(aa')(bb')(cc')^3b'$	$\Sigma(a_i\beta)^2(a'_i\beta')^2(a'\gamma')(a_i\beta')(\gamma\gamma')^2(_i\beta\delta')(\gamma\delta')\delta'$	$\{C_{10}f_{(1)}\}^3$
31	C_{24}	$(a'b')^2(a'c')(ac')^3(bb')^2ba'$	$\Sigma(a'_i\beta')^2(a'\gamma')(a_i\beta')(_i\beta\gamma')(\gamma\gamma')(a\delta')(_i\beta\delta')\gamma\delta'$	$\{C_{11}f_{(1)}\}^2$
32	F	$(a'b')^2(a'c')(ac')^3(ba')(bb')^2$	$\Sigma(a'_i\beta')^2(a'\gamma')(a_i\beta')(_i\beta\gamma')(\gamma\gamma')(a\delta')(_i\beta\delta')(\gamma\delta')$	$\{C_{11}f_{(1)}\}^3$
33	G	$(ab)^2(ac)(a'b')^2(a'c')(da')(bb')(db')(cc')^2(dc')$	$\Sigma(a_i\beta)^2(a\gamma)(a'_i\beta')^2(a'\gamma')(_i\beta\beta')(\gamma\gamma')^2(a\delta')(_i\beta\delta')(\gamma\delta')$	$\{C_{13}f_{(1)}\}^3$
34	C_{25}	$(ab)^2(ac)(de)^2(a'b')^2(a'c')(da')(bb')(eb')(cc')^2c'$	$\Sigma(a_i\beta)^4(a\gamma)(a'_i\beta')^2(a'\gamma')(_i\beta\beta')(\gamma\gamma')^2(\gamma\delta')^2\delta'$	$\{C_{13}H_{(1)}\}^2$
35	C_{26}	$(aa')^3(ba')(bb')(cb')^3b$	$\Sigma(a\alpha')^2(_i\beta\beta')^2(\gamma\gamma')^2(a\delta')(_i\beta\delta')\gamma$	$\{C_{14}p\}$
36	C_{27}	$(aa')^3(ba')(cd)^2(ce)(bb')(db')(eb')^2b$	$\Sigma(a_i\beta)^2(a\gamma)(a\alpha')(_i\beta a')(_i\beta\beta')(\gamma_i\beta')(\gamma\gamma')^2(a\delta')(_i\beta\delta')\gamma$	$\{C_{14}s\}$
37	C_{28}	$(a'b')^2(aa')^2(ab')(cb')(bc')^3(cc')c$	$\Sigma(a'_i\beta')^2(a\alpha')(_i\beta\beta')(a\gamma')(_i\beta\gamma')(\gamma\gamma')(a\delta')(\gamma\delta')^2{}_i\beta$	$\{C_{14}\pi\}$
38	C_{29}	$(ab)^2(cd)^2(a'b')^2(a'c')(aa')(bb')(cb')(dc')c'^2$	$\Sigma(a_i\beta)^4(a'_i\beta')^2(a'\gamma')(\gamma\gamma')^2(\gamma\delta')^2{}_i\beta\delta'$	$\{C_{10}H_{(1)}\}^2$
39	C_{30}	$(ab)^2(ac)(a'b')^2(a'c')(cb')^2(dc')^3ba'$	$\Sigma(a_i\beta)^2(a\gamma)(a'_i\beta')^2(a'\gamma')(a_i\beta')(_i\beta\gamma')(\gamma\gamma')(\gamma\delta')^2{}_i\beta\delta'$	$\{C_{11}J_{(1)}\}^2$

* A unique form.

TABLE X—Continued.

CUBIC AND QUARTIC.

No.	Name.	Aronhold and Clebsch Symbol.	Root Symbol.	Transvectant.
40	K	$(ab)^2(aa')(ca')^3(bb')(db')^3$	$\Sigma(\alpha_1\beta)^2(\alpha\alpha')^2(_1\beta_1\beta')^2(\gamma\gamma')^2(\gamma\delta')^2$	$\{qp\}$
41	E	$(ab)^2(aa')(ca')^3(de)^2(df)(bb')(eb')(_1f'b')^2$	$\Sigma(\alpha_1\beta)^4(\alpha\gamma)(\alpha\alpha')(_1\beta\alpha')(_1\beta_1\beta')(\gamma_1\beta')(\gamma\gamma')^2(\gamma\delta')^2$	$\{qs\}$
42	I	$(ab)^2(aa')(ca')^3(b'c')^2(db')^2(bc')(dc')$	$\Sigma(\alpha_1\beta)^2(\alpha'_1\beta')^2(\alpha\alpha')(_1\beta_1\beta')(\alpha\gamma')(_1\beta\gamma')(\gamma\gamma')(\gamma\delta')^3$	$\{q\pi\}$
43	L	$(ab)^2(aa')(ca')^3(de)^2(df)(b'c')^2(bb')^2(bc')(ec')$	$\Sigma(\alpha_1\beta)^4(\alpha\gamma)(\alpha'_1\beta')^2(\alpha\alpha')(_1\beta_1\beta')(\gamma\gamma')^3(_1\beta\delta')(\gamma\delta')^2$	$\{q\sigma\}$
44	M	$(aa')^3(ba')(bb')(cb')^3(bc')(dc')^3$	$\Sigma(\alpha\alpha')^3(\beta_1\beta')^3(\gamma\gamma')^3(\alpha\delta')(_1\beta\delta')(\gamma\delta')$	$\{C_{26}p\}$
45	N	$(aa')^3(ba')(bb')(cb')^3(bc')(de)^2(df)(ec')(_1fc')^2$	$\Sigma(\alpha_1\beta)^2(\alpha\gamma)(\alpha\alpha')^2(_1\beta\alpha')(_1\beta_1\beta')^2(\gamma_1\beta')(\gamma\gamma')^3(\alpha\delta')(_1\beta\delta')(\gamma\delta')$	$\{C_{26}s\}$
46	O	$(aa')^3(ba')(bb')(cb')^3(c'd')^2(dc')^2(bd')(dd')$	$\Sigma(\alpha'_1\beta')^2(\alpha\alpha')^2(_1\beta_1\beta')^2(\alpha\gamma')(_1\beta\gamma')(\gamma\gamma')^2(\alpha\delta')(_1\beta\delta')(\gamma\delta')^2$	$\{C_{26}\pi\}$
47	R	$(ab)^2(a'b')^2(ca')^2(cb)'(ab')(c'd')^2(ad')(dd')(dc')^2$	$\Sigma(\alpha_1\beta)^2(\alpha'_1\beta')^4(\alpha\gamma')^2(_1\beta\gamma')^2(\gamma\delta')^4$	$\{C_{19}\pi\}$
48	S	$(ab)^2(a'b')^2(ca')^2(cb')(ab')(de)^2(df)(c'd')^2(ad')(ed')(_1fc')^2$	$\Sigma(\alpha_1\beta)^4(\alpha\gamma)(\alpha'_1\beta')^4(\alpha\gamma')(_1\beta\gamma')(\gamma\gamma')^2(_1\beta\delta')(\gamma\delta')^3$	$\{C_{19}\sigma\}$
49	C_{31}	$(ab)^2(ac)(ba')(ca')^2(b'c')^2(ea')(db')^2(dc')(ec')e$	$\Sigma(\alpha_1\beta)^2(\alpha\gamma)(\alpha'_1\beta')^2(_1\beta\alpha')(\gamma_1\beta')(\alpha\gamma')(_1\beta\gamma')(\gamma\gamma')(\alpha\delta')(\gamma\delta')^2{}_1\beta$	$\{C_{18}s\}$
50	C_{32}	$(a'b')^2(aa')^2(ab')(bb')(c'd')^2(cc')^2(bd')(cd')b$	$\Sigma(\alpha'_1\beta')^4(\alpha\gamma')^2(_1\beta\gamma')^2(\alpha\delta')(\gamma\delta')^3{}_1\beta$	$\{C_{18}\pi\}$
51	C_{33}	$(ab)^2(ac)(a'b')^2(ca')^2(bb')(c'd')^2(dc')^2(dd')(ed')(eb')e$	$\Sigma(\alpha_1\beta)^2(\alpha\gamma)(\alpha'_1\beta')^4(\alpha\gamma')(_1\beta\gamma')(\gamma\gamma')^2(_1\beta\delta')^2(\gamma\delta')^2\alpha$	$\{C_{18}\sigma\}$
52	T	$(ab)^2(cd)^2(_ef)^2(a'b')^2(a'c')(aa')(bb')(cb')(dc')(ec')(_1fc')$	$\Sigma(\alpha_1\beta)^6(\alpha'_1\beta')^2(\alpha'\gamma')(\gamma_1\beta')(\gamma\gamma')^2(\gamma\delta')^3$	$\{C_{29}H_1\}^2$
53	U	$(aa')^3(ba')(cd)^2(ce)(bb')^2(db')(eb')(c'd')^2(_1fc')^2(bd')(_1fd')$	$\Sigma(\alpha_1\beta)^2(\alpha\gamma)(\alpha'_1\beta')^2(\alpha\alpha')(_1\beta\alpha')(\beta_1\beta')(\gamma_1\beta')(\alpha\gamma')(_1\beta\gamma')(\gamma\gamma')^2(\alpha\delta')(_1\beta\delta')(\gamma\delta')^2$	$\{C_{27}\pi\}$
54	V	$(a'b')^2(aa')^2(ab')(cb')(bc')^3(cc')(d'e)^2(dd')^2(ce')(de')$	$\Sigma(\alpha'_1\beta')^4(\alpha\alpha')(_1\beta_1\beta')(\alpha\gamma')^2(_1\beta\gamma')^2(\gamma\gamma')(\alpha\delta')(_1\beta\delta')(\gamma\delta')^3$	$\{C_{28}\pi\}$
55	W	$(a'b')^2(aa')^2(ab')(bb')(c'd')^2(cc')^2(bd')(cd')(de)^2(df)(be')(ee')(_1fe')^2$	$\Sigma(\alpha_1\beta)^2(\alpha\gamma)(\alpha'_1\beta')^4(_1\beta\alpha')(\gamma_1\beta')(\alpha\gamma')^2(_1\beta\gamma')^2(\gamma\gamma')(\alpha\delta')(_1\beta\delta')(\gamma\delta')^3$	$\{C_{32}s\}$

ERRATA *in the paper on* CONCOMITANT BINARY FORMS IN TERMS OF THE ROOTS *which appeared in Vol. IX.*

P. 96, l. 22. Substitute *four* for *five*.

P. 107, l. 20. Substitute *covariant* for *invariant*.

Pp. 110–114. An error in the subject-matter of Arts. 15–18 may be corrected by substituting *unique** for *irreducible* in l. 11 and l. 15, p. 110, and in l. 3, p. 112, and substituting *that is* for *and* in l. 12, p. 110; and by omitting *irreducible* in l. 12, p. 110, l. 14, p. 111, l. 27, p. 112, and in l. 20, p. 114, and both *an irreducible form* and the following word *irreducible* in l. 9, p. 113.

P. 116, Art. 21. Substitute *form-systems* for *form system* in l. 1, and *four* for *five* in l. 3.

P. 118, l. 13. Substitute $(\partial\varepsilon)^2$ for $(\partial\varepsilon)$.

P. 119, l. 5. Substitute $-$ for $+$.

P. 124, l. 18. Insert $\begin{bmatrix}\beta_2 & a_2\\ \beta_1 & a_1\end{bmatrix}^{e_{12}}\begin{bmatrix}\gamma_2 & \beta_2\\ \gamma_1 & \beta_1\end{bmatrix}^{e_{23}}\ldots\begin{bmatrix}x & a_2\\ y & a_1\end{bmatrix}^{e_1}\ldots$ after Σ.

P. 124, l. 23. Omit *a covariant, and hence*.

P. 125, l. 8. Substitute *the summation* for Σ.

P. 127, l. 17 and 18. Change θ to ϑ.

P. 140, l. 18. Invert last fraction.

P. 146, l. 2. Omit *fixed*. l. 13. Insert 57 as no. of Art.

Pp. 155–6, References. Substitute Faà for F. F.; and Hilbert for Hibbert.

* See the preceding paper for a definition of a *unique form*.

APPROXIMATE METHODS FOR THE TREATMENT OF DIFFERENTIAL EQUATIONS.

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Various approximate rules of integration devised for the purpose of dealing with functions which do not admit of expression in analytical form are well known. It is the purpose of the present paper to call attention to various allied methods for the approximate treatment of such differential equations as do not admit of treatment by the usual analytical methods. For purposes of illustration only two of the familiar rules for integration will be used.

Let $y_0 y_1$, etc., denote consecutive values of the function separated by equal increments Δx. Then by the trapezoidal rule the integral of the function for one interval is :—

$$A_0' = \frac{\Delta x}{2}(y_0 + y_1),$$

while by the so-called Simpson's one-third, or Stirling's rule, the integral for two intervals is :—

$$A_0^2 = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2).$$

If for convenience we denote successive derivatives with reference to x by $y'\ y''\ y'''$, etc., we shall have from the trapezoidal rule :—

$$y_1 = y_0 + \frac{\Delta x}{2}(y_0' + y_1') \tag{1}$$

$$y_1' = y_0' + \frac{\Delta x}{2}(y_0'' + y_1''), \tag{2}$$

or by substituting from (2) in (1)

$$y_1 = y_0 + \Delta x y_0' + \frac{\overline{\Delta x}^2}{4}(y_0'' + y_1''). \tag{3}$$

Similarly for Simpson's rule we have :—

$$y_2 = y_1 + \frac{\Delta x}{3}(y_0' + 4y_1' + y_2') \tag{4}$$

$$y_2' = y_0' + \frac{\Delta x}{3}(y_0'' + 4y_1'' + y_2''), \tag{5}$$

or by substitution from (5) in (4)

$$y_2 = y_0 + \frac{\Delta x}{3}(2y_0' + 4y_1') + \frac{\overline{\Delta x}^2}{9}(y_0'' + 4y_1'' + y_2''). \tag{6}$$

From Maclaurin's theorem we may also deduce similar expressions which may be used in like manner, though they are relatively somewhat less accurate than those derived above. For an increment Δx we thus have :—

$$y_1 = y_0 + y_0'\Delta x + \tfrac{1}{2}\, y_0''\overline{\Delta x}^2 + \tfrac{1}{6}\, y_0'''\overline{\Delta x}^3 + \&c., \tag{7}$$

$$y_1' = y_0' + y_0''\Delta x + \tfrac{1}{2}\, y_0'''\overline{\Delta x}^2 + \&c., \tag{8}$$

$$y_1'' = y_0'' + y_0'''\Delta x + \&c. \tag{9}$$

Now taking any differential equation, of the second order for example, suppose that we have given the particular values at a starting point which we may denote by subscript 0. Denoting an adjacent point by subscript 1, let us substitute in the equation the values of y_1' and y_1 as given in (2) and (3). The resulting equation will then express y_1'' as a function of y_0 y_0' and y_0'' and will be of degree equal to that of the term of highest degree involving y or its derivatives. Hence if the original equation is linear the resulting value of y_1'' is readily found and this substituted in (2) and (1) will give values of y_1' and y_1 which will necessarily satisfy the differential equation at this point. The point (1) being thus determined, it is in turn considered as known and the next adjacent point is determined in a similar manner. In this way by a step by step process the values of y are found for a series of points covering the range for which the history of the function is desired. If the equation is linear it is usually preferable to solve analytically for y_1'' and then substitute in this solution the known values, thus determining the successive values of y_1'' and thence those of y_1' and y_1. If the equation is not linear, special means suitable to the case may be adopted to effect the solution. In such cases, particularly where transcendental functions are involved, it may be necessary to treat the equation by tentative and approximate methods rather than by direct solution. To this end a value may be assumed for y_1'' and from (1) (2) and (3) the resultant approximate values of y_1' and y_1 found. These values of y_1' and y_1 being substituted in the differential equation, the latter may then be solved for y_1'' by such means as may be appropriate or necessary. A comparison of this result with the assumed value will furnish a basis for the correction of the latter, and thus by a few trials a set of values may be found which will satisfy all conditions.

It will be noted that the essential feature of the process is the substitution of an algebraic equation for the original differential equation, and the solution of the latter is thus made to depend upon that of the former.

If two values of the original function are known, one by approximate determination as above for example, then the values of y_2' and y_2 as given in

(5) and (6) may be substituted and the equation solved for y_2''. This being substituted back will give values of y_2' and y_2, and thus the next point becomes known. In like manner the values for points (1) and (2) may be used to determine point (3), and so forward over the range for which the function is desired.

The values of y_1 y_1' y_1'', etc., given in (7) (8) and (9) may if desired be used is the same general manner as those above considered.

The errors involved in these operations will be those due to the departure of the function treated by the rules for approximate integration, from forms for which such rules are accurate. Such errors may be reduced to any desired amount by taking the values of Δx sufficiently small. Some increase in accuracy may also be gained by the use of rules of higher orders. In general, however, it will be found preferable in all approximate integrations to limit the rules to those of the first and second orders as given above, and to control the question of resultant error by an appropriate selection of interval Δx. The trapezoidal or rule of the first order assumes between points 0–1, 1–2, etc., a linear law, and will therefore require a relatively close spacing of ordinates.

The Simpson's rule or rule of the second order assumes between points 0–2, 1–3, etc., a second degree parabolic law. With spacing the same as for the trapezoidal rule the accuracy will be considerably greater, or the same accuracy may be attained with a wider spacing or greater value of Δx. By remembering the nature of the error for the trapezoidal rule, and noting whether the functions y'' and y' are increasing or decreasing, convex or concave to x, the tendency of the errors due to this rule may be readily ascertained.

If certain quantities in the original differential equation are given graphically as by a curve for which no analytical equation is known, the methods above outlined are applicable without change, and one of the chief fields of usefulness for such methods will be found in the treatment of equations of this character. Further reference to such application will be found in the examples given below.

As with the treatment of differential equations in general, special manipulation or transformation may be useful in particular cases, but no general rules can be given for such cases, and the broad outlines as given above will serve to indicate the general line of treatment when the solution is to be effected by these methods.

The general application of the methods herein presented is evidently independent of the order of the equation, and except for the greater amount of work involved in carrying out the operations, those of the third or fourth or of higher orders are quite as readily treated as those of the first or second.

These methods may also be sometimes used to advantage with simultaneous differential equations when the attempt at elimination would result in an equation too complicated for treatment in the usual manner. An illustration is found in the examples given below.

Differential equations involving imaginaries or complex quantities are as readily treated by these methods as are any other. That is, the existence of complex or imaginary quantities gives rise to no additional difficulty of treatment beyond the added work required to carry out the operations involved.

Except in particular cases these methods do not seem likely to be of much aid in the treatment of partial differential equations. The quantities are here so related that usually a general solution must first be found, and then made particular by a substitution of boundary or other conditions, and this is not possible by a step by step process such as that here described.

The work involved in the determination of an unknown from a differential equation by the methods above outlined will depend wholly on circumstances. In many cases it might be prohibitive, in others not. It will depend on the nature of the functions involved, on the accuracy desired, and on the range to be covered between the known and the unknown. The question of the availability of such methods will then depend on these facts, and on the time available and justifiable for the determination of the value desired. The same considerations affect the final accuracy which may be made whatever is desired by the means already pointed out. The present paper is not concerned with the question of the length or shortness of the operations involved, but rather with the development of means whereby results may be obtained if their work will justify the labor necessary.

Illustrations.

As an example made up simply to show the application of the method to an equation too complicated for solution by analytical means, take the following:—

$$\left[y\frac{dy}{dx}\frac{d^2y}{dx^2}\right] = \sin\left[x\frac{dy}{dx}\right] + \log\left[y\frac{dy}{dx}\right]$$

with the initial conditions $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 0$. This equation has been treated by the above method with the trapezoidal rule, taking $\Delta x = .2$, between the limits $x = 0$ and $x = 3$. The graphical histories of y and its two derivatives are given in Fig. 1.

To show the error in a given case the method was applied to a Bessel's equation of zero order, and the value of $J_0(x)$ was found from $x = 0$ to $x = 1$ by the trapezoidal rule and with intervals of .1. The resultant value of y_1'' in terms of known quantities is in this case readily seen to be :—

$$y_1'' = -\frac{\left\{y_0' + \frac{\Delta x}{2} y_0''\right\} + x\left\{y_0 + \Delta x y_0' + \frac{\overline{\Delta x}^2}{4} y_0''\right\}}{\left\{x + \frac{\Delta x}{2} + x\frac{\overline{\Delta x}^2}{4}\right\}}.$$

In the following table the first column gives the values thus found, and the second, the true values as found from the series in the usual way :—

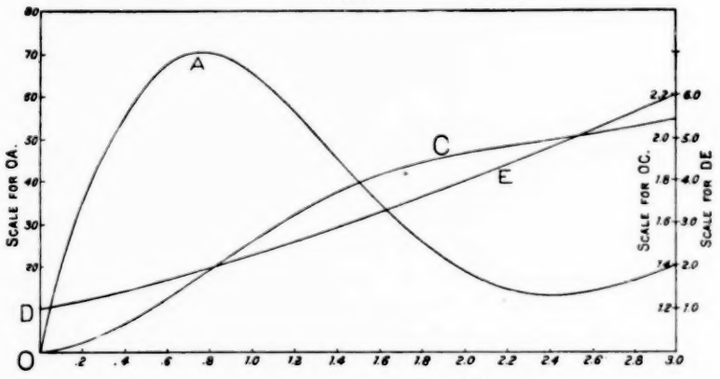
x	$J_0(x)$	
0	1.0000	1.0000
.1	.9975	.9975
.2	.9903	.9903
.3	.9776	.9776
.4	.9604	.9604
.5	.9385	.9385
.6	.9121	.9120
.7	.8813	.8812
.8	.8464	.8463
.9	.8077	.8075
1.0	.7654	.7652

In the case of the common pendulum giving rise to the equation :—

$$\frac{d^2\theta}{dt^2} + \frac{g}{e}\sin\theta = 0$$

a well known elliptic integral furnishes the solution. The evaluation of this integral fails however when the initial displacement is very large on account of the slow convergence of the series. Thus let the initial displacement be 3.1406 or .001 less than π. The series in such case is of no avail and other means must be sought. By the methods above described the period of oscillation for a seconds pendulum is found in such case to be 5.725 sec.

FIG. 1.

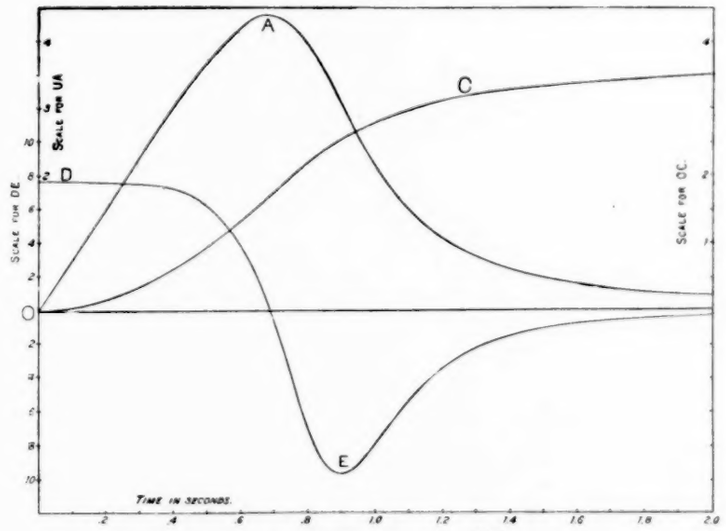


OA gives history of $\frac{d^2y}{dx^2}$

OC " " " $\frac{dy}{dx}$

DE " " " y

FIG. 2.



DE gives history of $\frac{d^2\theta}{dt^2}$

OA " " " $\frac{d\theta}{dt}$

OC " " " θ

As a test of accuracy the period for an initial displacement of 30° was computed by the same means and compared with the result given by the series. The first gave 1.017 sec. and the second 1.0174. This was by the trapezoidal rule and $\Delta t = .1$ sec.

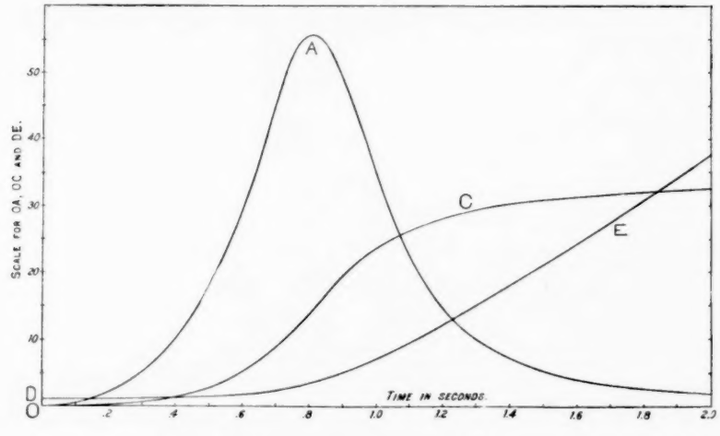
Given the following problem. A vertical rod is provided with a horizontal arm indefinite in extent, on which may slide without friction a weight m located initially at a distance a from the axis of the rod. Let the vertical rod be urged to rotate about its own axis by a constant torque such as would be produced by a falling weight W acting by means of a cord led over a fixed pulley to a drum attached to the rod. It is required to find the motion of the weight m. This gives rise to the following simultaneous equations :—

$$\frac{d^2r}{dt^2} = r\left[\frac{d\theta}{dt}\right]^2$$

$$g\,W = a\,(W + B)\frac{d^2\theta}{dt^2} + \frac{mr}{a}\left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right].$$

Their solution by analytical means seems very difficult. By the approximate means above outlined, however, the problem may be solved and the history of the various functions derived. These are illustrated in Figs. 2 and 3 for the following numerical values: $W = 10$, $B = 10$, $a = 2$, $r_1 = 1$, $m = 4$.

FIG. 3.



OA gives history of $\frac{d^2r}{dt^2}$

OC " " " $\frac{dr}{dt}$

DE " " " r

In the case of an electric circuit without iron acted on by an alternating electro-motive force, the general equation is :—

$$f(t) = Ri + L\frac{\partial i}{\partial t}$$

where $f(t)$ represents the history of the electro-motive force and gives simply the successive ordinates of a geometrical curve, the characteristics of which may be found experimentally, but of which no equation is known. This represents an important class of problems in which a portion of the data is graphical and not algebraical. The solution by the method here presented is simple and very satisfactory, and from its special interest to engineering students has been given by the author elsewhere.*

Illustrative examples might be widely multiplied, but enough has been given to show the application of the methods herein developed, and the possibility of obtaining satisfactory practical results in cases where the equations are beyond the reach of the usual analytical methods.

* Sibley Journal of Engineering, Feb., 1897.

THE GENERATION OF SURFACES.

By Prof. H. A. Sayre, Collegeville, Pa.

Introduction.

1. A surface may be defined by a property common to each of its points; in this case we obtain the equation of the surface by translating this property analytically. But more usually, however, a surface is defined by the movement of a line in space. Suppose

$$f(x, y, z, c) = 0, \quad \varphi(x, y, z, c) = 0,$$

the equations of a line containing an arbitrary parameter c; if c varies in a continuous manner, the line moves in space and generates a surface. The equation of the surface is obtained by eliminating c between the equations of the moving line. In this investigation, a surface is supposed to be generated by a variable plane curve which turns around a line of its plane. It will be convenient to define the motion of the plane curve, by subjecting it to the condition of moving on fixed curves, which may be either plane or twisted. The generating curve in any position will be defined by means of two rectangular co-ordinates (z, t). Any surface may be generated in this way and surfaces generated by curves of the same kind have equations of the same form. Thus it will be shown, if the axis of z is taken for the axis of rotation, the generating curve being a straight line, that the equation of the surface will be of the form

$$z = x\varphi\left[\frac{y}{x}\right] + \varphi'\left[\frac{y}{x}\right].$$

If the generating curve is a circle the equation of the surface takes the form

$$x^2 + y^2 + z^2 + z\varphi_1\left[\frac{y}{x}\right] + x\varphi_2\left[\frac{y}{x}\right] + \varphi_3\left[\frac{y}{x}\right] = 0.$$

2. As far as I know, only one special case of the method has been used heretofore and that is the case of surfaces of revolution. A surface of revolution may be generated by the rotation of an invariable plane curve around a fixed right line lying in its plane. During the rotation the mutual positions of the line and curve remain unchanged. In order to find the equation of a surface of revolution, let us take for the axis of rotation the axis of z, and for the plane in which the generating curve was originally placed the (x, z) plane

of a system of rectangular co-ordinates. Suppose that the equation of the generating curve is

$$z = f(x).$$

Now imagine the surface cut by any plane which passes through the axis of z and assume in this plane a system of rectangular co-ordinates (z, t). Since this plane must cut out from the surface one of the original congruent curves, the equation of the section must be

$$z = f(t).$$

Now for all points of an arbitrary meridian we have evidently

$$t^2 = x^2 + y^2,$$

therefore

$$z = f(\sqrt{x^2 + y^2})$$

is the equation of the surface of revolution.

3. The method depends upon the properties of a system of homothetic plane curves. The special case of surfaces of revolution depends upon the properties of a system of homothetic concentric circles, the axis of the surface being perpendicular to the plane of the circles and passing through their common centre.

4. The same method may be applied to the study of twisted curves. For if two plane curves lying in the same plane are referred to the (z, t) co-ordinates, their intersections are referred to the same co-ordinates and will trace out a curve as the plane rotates about a line as an axis.

I.

General Theory.

1. Being given any system of homothetic curves in the xy plane, it may be represented by the equation

$$f(kx, ky) = 0, \tag{1}$$

the homothetic centre being at the origin. The equation

$$f(x, y) = 0$$

represents a curve of the system which will be called the fundamental curve and the curve

$$f(kx, ky) = 0$$

will be referred to as the curve (k). If we denote by ρ the absolute value of

the distance from the origin to a point P of the fundamental curve and by t the length of the radius vector (positive or negative according as it has the same sense as OP or the opposite sense) to the homologous point P' of the curve (k) we shall have

$$k = \frac{\rho}{t}. \tag{2}$$

Now suppose that at every point of these curves a perpendicular z is erected to the plane xy; the extremities of these perpendiculars trace out a surface. If z is determined for the point (x, y) of the curve (k) by

$$\psi\left[z, \frac{1}{k}, \frac{y}{x}\right] = 0, \tag{3}$$

the equation of the surface will be obtained by eliminating k between equations (1) and (3). For a fixed value of $\frac{y}{x}$, that is for the points of the surface obtained along a line TOT' drawn through the origin, the points of the surface satisfy $\left[\text{since } \frac{1}{k} = \frac{t}{\rho}\right]$ the equation

$$\psi\left[z, \frac{t}{\rho}, \frac{y}{x}\right] = 0 \tag{4}$$

ρ and $\frac{y}{x}$ being constant. This equation, considering z and t as rectangular coordinates, is the equation of the section of the surface by the plane ZOT. If

$$f(x, y) = 0$$

is an algebraic curve of order n, then upon any line through the origin we get n points P and n values of ρ, say $\rho_1, \rho_2, \ldots \rho_n$. To each of these values will correspond a curve ψ, accordingly the section of the surface by a plane through the axis of z will consist of the n plane curves (which may or may not be different)

$$\psi\left[z, \frac{t}{\rho_i}, \frac{y}{x}\right] = 0, \qquad i = 1, 2, \ldots, n$$

t is not measured necessarily in the same direction in all these equations. For those of the points P which lie on OT, OT is the positive direction for t, for those which lie on OT', OT' is the positive direction. The surface obtained by eliminating k between

$$f(kx, ky) = 0,$$
$$\psi\left[z, \frac{1}{k}, \frac{y}{x}\right] = 0$$

is generated therefore by the motion of these n plane curves as their common plane rotates about the axis of z. The equation

$$\psi\left[z, \frac{t}{\rho}, \frac{y}{x}\right] = 0$$

will be called the type equation of the generating plane curves. If the homothetic centre is at an ordinary point of the fundamental curve, k will enter the equation of the homothetic system to the $n - 1$ degree and the section of the surface (leaving aside for the moment the special position of the cutting plane where TOT' is tangent to the fundamental curve at the origin) will consist of the $n - 1$ curves

$$\psi\left[z, \frac{t}{\rho_i}, \frac{y}{x}\right] = 0, \quad \rho_i \neq 0, \qquad i = 2, 3, \ldots, n$$

and the curve

$$\psi\left[z, \frac{t}{\rho_1}, \frac{y}{x}\right] = 0, \quad \rho_1 = 0.$$

When TOT' is tangent to the fundamental curve at the origin, the section will consist of

$$\psi\left[z, \frac{t}{\rho_i}, \frac{y}{x}\right] = 0, \quad \rho_i \neq 0, \qquad i = 3, 4, \ldots, n$$

and

$$\psi\left[z, \frac{t}{\rho_i}, \frac{y}{x}\right] = 0, \quad \rho_i = 0. \qquad i = 1, 2$$

If the homothetic centre is at a multiple point of order m the section will consist of the curves

$$\psi\left[z, \frac{t}{\rho_i}, \frac{y}{x}\right] = 0, \quad \rho_i \neq 0 \quad i = m+1, m+2, \ldots, n$$

$$\psi\left[z, \frac{t}{\rho_i}, \frac{y}{x}\right] = 0, \quad \rho_i = 0. \qquad i = 1, 2, \ldots, m$$

When TOT' is tangent to the fundamental curve at the origin the section will consist of the curves

$$\psi\left[z, \frac{t}{\rho_i}, \frac{y}{x}\right] = 0, \quad \rho_i \neq 0, \quad i = m+2, m+3, \ldots, n$$

$$\psi\left[z, \frac{t}{\rho_i}, \frac{y}{x}\right] = 0, \quad \rho_i = 0. \qquad i = 1, 2, \ldots, m+1$$

2. If we desire the line around which the plane of the generating curves

rotates and which cuts the plane of xy in the homothetic centre to be parallel to the axis of z, it is sufficient to eliminate k between

$$f[kx + (1-k)x_0,\ ky + (1-k)y_0] = 0\,,$$

$$\psi\left[z, \frac{1}{k}, \frac{y-y_0}{x-x_0}\right] = 0\,,$$

x_0, y_0 being the coordinates of the homothetic centre.

3. Let us consider the circle

$$x^2 + y^2 = r^2$$

as the fundamental curve. The system of homothetic curves will be

$$k^2x^2 + k^2y^2 = r^2$$

from which we obtain

$$\frac{t^2}{\rho^2} = \frac{x^2 + y^2}{r^2}\,.$$

Let the type equation be denoted by

$$\psi\left[z, \frac{t}{\rho}, \frac{y}{x}\right] = 0\,.$$

Since the line TOT' cuts the fundamental circle in two points which are symmetrical with respect to the origin O, we have

$$\rho_1 = \rho_2$$

whence the equations

$$\psi\left[z, \frac{t}{\rho_1}, \frac{y}{x}\right] = 0\,,$$

$$\psi\left[z, \frac{t}{\rho_2}, \frac{y}{x}\right] = 0\,,$$

represent two curves lying in the plane ZOT and these curves are symmetrical with respect to the axis of z. These two curves represent the section cut out by the plane ZOT from the surface obtained by eliminating $\frac{t}{\rho}$ between

$$\psi\left[z, \frac{t}{\rho}, \frac{y}{x}\right] = 0\,,$$

$$\frac{t}{\rho} = \pm\frac{\sqrt{x^2 + y^2}}{r}\,.$$

If $\frac{y}{x}$ is actually present in the equation of the type curve, the section will

vary with the cutting plane. If the surface is a surface of revolution, the section does not change with the cutting plane. This is indicated by the absence of $\frac{y}{x}$ from the type equation. Since the circle is a curve of the second order, there are two generating curves which may or may not coincide. If the type equation

$$\phi\left[z, \frac{t}{\rho}, \frac{y}{x}\right] = 0$$

contains only even powers of t, the two generating curves will be coincident and this curve will be symmetrical with respect to the axis of z.

4. Assuming that the equations of the sections of any proposed surface by planes through the axis of z can be represented by one and the same equation involving an arbitrary parameter determining the position of the section, there is no loss of generality in writing this equation

$$\phi\left[z, \frac{t}{\rho}, \frac{y}{x}\right] = 0,$$

since $\frac{y}{x}$ can be taken as that parameter. Accordingly by what precedes, any surface, whose sections by planes through the axis of z are symmetrical with respect to this axis, can be represented in the foregoing manner by means of a system of concentric circles and thus have its equation presented in the form

$$\phi\left[z, \frac{\sqrt{x^2 + y^2}}{r}, \frac{y}{x}\right] = 0.$$

In the case of a surface in which the sections are not symmetrical with respect of the axis of z, the same method would be applicable, but would give not only the proposed surface, but also the symmetrical surface with respect to the axis of z. It may happen that apart from the occurrence of $\sqrt{x^2 + y^2}$ through the value of t, it may be brought in also through the parameter $\frac{y}{x}$. In this event the rationalization of the equation may take place through cancellation of the radical and without raising to powers. Then since the double sign must be understood in the value of $t = \sqrt{x^2 + y^2}$, one surface will be represented when one sign is taken, the symmetrical surface when the other sign is taken.

5. As an instance of this breaking up into two distinct surfaces let us generate the plane

$$x + y + z - 1 = 0,$$

by the motion of a straight line which passes through the point (0, 0, 1) on the axis of z. For the type equation of the generating curve let us take

$$z - 1 = mt,$$

m being the tangent of the angle which the line makes with the radius vector OT. We shall suppose m to vary with the position of the (z, t) plane, and accordingly to be a function of $\frac{y}{x}$, viz., we shall suppose

$$m = \frac{x + y}{\sqrt{x^2 + y^2}}$$

so that the type equation is

$$z - 1 = \frac{x + y}{\sqrt{x^2 + y^2}}\, t\,.$$

To obtain the equation of the surface we must substitute for t in the preceding equation its value $\pm \sqrt{x^2 + y^2}$. The equation then breaks up into the two equations

$$x + y + z - 1 = 0,$$

$$z - x - y - 1 = 0.$$

These two equations represent the proposed plane and its symmetrical plane. To generate a surface of revolution we must eliminate t between the equation of the type curve

$$\phi\,(z, t) = 0$$

and

$$t = \sqrt{x^2 + y^2}.$$

If only even powers of t occur in the type equation the two generating curves will coincide and only one surface will be generated. If odd powers of t occur there is nothing to make the irrationalities disappear without raising to powers, for the type equation does not contain $\frac{y}{x}$. We see therefore that the two sheets generated by invariable curves, whose positions with respect to the axis of z do not change, belong to the same surface.

6. A point on a twisted curve will be known if its z is known and if its projection on the plane of xy is known. We may therefore represent the equations of a twisted curve by

$$z = g\left[\frac{y}{x}\right],$$

$$\frac{t}{\rho} = h\left[\frac{y}{x}\right].$$

The first equation represents a right conoid having the axis of z as an axis, while the second represents a cylinder having its generators parallel to the axis of z. The second equation will be called the type equation. If for the fundamental curve we take the circle

$$x^2 + y^2 = \rho^2,$$

the equations of the twisted curve become

$$z = g\left[\frac{y}{x}\right],$$

$$t = h\left[\frac{y}{x}\right].$$

The type equation

$$t = h\left[\frac{y}{x}\right]$$

may be considered as representing a curve in the plane of xy which is symmetrical with respect to the origin. In the same way as in the case of surfaces, we observe that this curve may break up into a curve which is not symmetrical with respect to the origin together with the symmetric of this curve. Assuming that the points of any twisted curve cut out by planes through the axis of z can be represented by the same equations involving an arbitrary parameter determining the position of the cutting plane, there is no loss of generality in writing these equations

$$z = g\left[\frac{y}{x}\right],$$

$$t = h\left[\frac{y}{x}\right],$$

since $\frac{y}{x}$ can be taken as that parameter. Accordingly, by what precedes, a curve whose points of section by planes through the axis of z are symmetrical with respect to this axis may be represented in the foregoing manner. In the case of a curve in which the points of section are not symmetrical with respect to the axis of z, the same method would be applicable, but would give not only the proposed curve, but also its symmetric with respect to the axis of z.

7. A twisted curve may be represented also by means of the equations

$$\varphi\left[z, \frac{t}{\rho}, \frac{y}{x}\right] = 0,$$

$$\psi\left[z, \frac{t}{\rho}, \frac{y}{x}\right] = 0.$$

In the plane of ZOT these equations, taken together, will represent a certain number of points which are determined as the intersections of plane curves. As the plane rotates about the axis of z these points will trace out a curve.

II.

The Circle as the Fundamental Curve.

1. Problem. The common plane of three circles passes through the axis of z and each circle moves on three arbitrary curves. Determine the surfaces generated by these circles, the surfaces traced out by their radical axes, and the surface generated by the circle which cuts the three circles orthogonally. Determine also the curves on which the centres of the given circles move and the curves described by their radical centre. The three given circles will be referred to as the circles s_1, s_2, s_3. Suppose that the equation of s_1 is

$$z^2 + t^2 + Az + Bt + C = 0,$$

where A, B, C are to be determined by the condition that the circle passes through the points (z_1, t_1), (z_2, t_2), (z_3, t_3). If these points lie on the curves

$$z = g_1\left[\frac{y}{x}\right], \qquad t = h_1\left[\frac{y}{x}\right];$$

$$z = g_2\left[\frac{y}{x}\right], \qquad t = h_2\left[\frac{y}{x}\right];$$

$$z = g_3\left[\frac{y}{x}\right], \qquad t = h_3\left[\frac{y}{x}\right],$$

respectively, the equation of the circle s_1 is

$$\begin{vmatrix} z^2 + t^2 & z & t & 1 \\ g_1^2 + h_1^2 & g_1 & h_1 & 1 \\ g_2^2 + h_2^2 & g_2 & h_2 & 1 \\ g_3^2 + h_3^2 & g_3 & h_3 & 1 \end{vmatrix} = 0,$$

which may be written

$$z^2 + t^2 + 2zD_1\left[\frac{y}{x}\right] + 2tE_1\left[\frac{y}{x}\right] + F_1\left[\frac{y}{x}\right] = 0.$$

The equations of the three circles may therefore be written

$$\left.\begin{aligned} s_1 \quad & z^2 + t^2 + 2zD_1\begin{Bmatrix} y \\ x \end{Bmatrix} + 2tE_1\begin{Bmatrix} y \\ x \end{Bmatrix} + F_1\begin{Bmatrix} y \\ x \end{Bmatrix} = 0\,, \\ s_2 \quad & z^2 + t^2 + 2zD_2\begin{Bmatrix} y \\ x \end{Bmatrix} + 2tE_2\begin{Bmatrix} y \\ x \end{Bmatrix} + F_2\begin{Bmatrix} y \\ x \end{Bmatrix} = 0\,, \\ s_3 \quad & z^2 + t^2 + 2zD_3\begin{Bmatrix} y \\ x \end{Bmatrix} + 2tE_3\begin{Bmatrix} y \\ x \end{Bmatrix} + F_3\begin{Bmatrix} y \\ x \end{Bmatrix} = 0\,. \end{aligned}\right\} \qquad \text{(A)}$$

The ordinary method for the determination of a circle orthogonal to three given circles is now applicable and gives for its equation

$$\begin{vmatrix} D_1 + z & E_1 + t & zD_1 + tE_1 + F_1 \\ D_2 + z & E_2 + t & zD_2 + tE_2 + F_2 \\ D_3 + z & E_3 + t & zD_3 + tE_3 + F_3 \end{vmatrix} = 0\,.$$

This circle will be called the circle O. To obtain the equations of the surfaces generated by the circles s_1, s_2, s_3, O, we must substitute $\pm\sqrt{x^2+y^2}$ for t in the equations of the circles. Doing this we get for the equations of the surfaces

$$\begin{aligned} s_1 \quad & z^2 + x^2 + y^2 + 2zD_1 + 2E_1\sqrt{x^2+y^2} + F_1 = 0\,, \\ s_2 \quad & z^2 + x^2 + y^2 + 2zD_2 + 2E_2\sqrt{x^2+y^2} + F_2 = 0\,, \\ s_3 \quad & z^2 + x^2 + y^2 + 2zD_3 + 2E_3\sqrt{x^2+y^2} + F_3 = 0\,, \end{aligned}$$

$$O \quad \begin{vmatrix} D_1 + z & E_1 \pm \sqrt{x^2+y^2} & zD_1 \pm \sqrt{x^2+y^2}\,E_1 + F_1 \\ D_2 + z & E_2 \pm \sqrt{x^2+y^2} & zD_2 \pm \sqrt{x^2+y^2}\,E_2 + F_2 \\ D_3 + z & E_3 \pm \sqrt{x^2+y^2} & zD_3 \pm \sqrt{x^2+y^2}\,E_3 + F_3 \end{vmatrix} = 0\,.$$

For the surfaces traced out by the radical axes of the three circles we have

$$\begin{aligned} s_1s_2 \quad & 2z(D_1 - D_2) + 2\sqrt{x^2+y^2}\,(E_1 - E_2) + F_1 - F_2 = 0\,, \\ s_2s_3 \quad & 2z(D_2 - D_3) + 2\sqrt{x^2+y^2}\,(E_2 - E_3) + F_2 - F_3 = 0\,, \\ s_3s_1 \quad & 2z(D_3 - D_1) + 2\sqrt{x^2+y^2}\,(E_3 - E_1) + F_3 - F_1 = 0\,. \end{aligned}$$

Writing the equations of the three circles in the form

$$s_1 \quad (z + D_1)^2 + (t + E_1)^2 = D_1^2 + E_1^2 - F_1,$$
$$s_2 \quad (z + D_2)^2 + (t + E_2)^2 = D_2^2 + E_2^2 - F_2,$$
$$s_3 \quad (z + D_3)^2 + (t + E_3)^2 = D_3^2 + E_3^2 - F_3,$$

we see that their centres will trace out the curves

$$s_1 \quad z = -D_1, \quad t = -E_1;$$
$$s_2 \quad z = -D_2, \quad t = -E_2;$$
$$s_3 \quad z = -D_3, \quad t = -E_3.$$

Using equations (A) we obtain for the equations of the curve traced out by their radical centre

$$z = -\tfrac{1}{2}\frac{\begin{vmatrix} F_2 - F_1 & E_2 - E_1 \\ F_3 - F_2 & E_3 - E_2 \end{vmatrix}}{\begin{vmatrix} D_1 - D_2 & E_1 - E_2 \\ D_2 - D_3 & E_2 - E_3 \end{vmatrix}}, \quad t = -\tfrac{1}{2}\frac{\begin{vmatrix} D_1 - D_2 & F_1 - F_2 \\ D_2 - D_3 & F_2 - F_3 \end{vmatrix}}{\begin{vmatrix} D_1 - D_2 & E_1 - E_2 \\ D_2 - D_3 & E_2 - E_3 \end{vmatrix}}.$$

2. Ruled Surfaces. Determine the equation of the surface generated by the motion of a straight line which meets the axis of z and moves on two arbitrary curves.

The type equations may be written

$$\frac{z - z_0}{t - t_0} = \frac{z_1 - z_0}{t_1 - t_0}.$$

Suppose that the point (z_0, t_0) lies on the curve

$$z = g_0\left[\frac{y}{x}\right], \quad t = h_0\left[\frac{y}{x}\right],$$

the point (z_1, t_1) on

$$z = g_1\left[\frac{y}{x}\right], \quad t = h_1\left[\frac{y}{x}\right].$$

The type equation now becomes

$$\frac{z - g_0}{t - h_0} = \frac{g_1 - g_0}{h_1 - h_0} = F\left[\frac{y}{x}\right] \text{ say.}$$

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